Reconstruction from projections

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5.2.2003

Outline

Image reconstruction

Radon transform

Fourier slice theorem

Inverse Radon transform by FBP

Iterative reconstruction

- Penalized iterative reconstruction

Conclusion

Image reconstruction

- The image is not seen by the scanner, but it's projection profiles are measured.

Measured PET sinogram data Reconstructed image

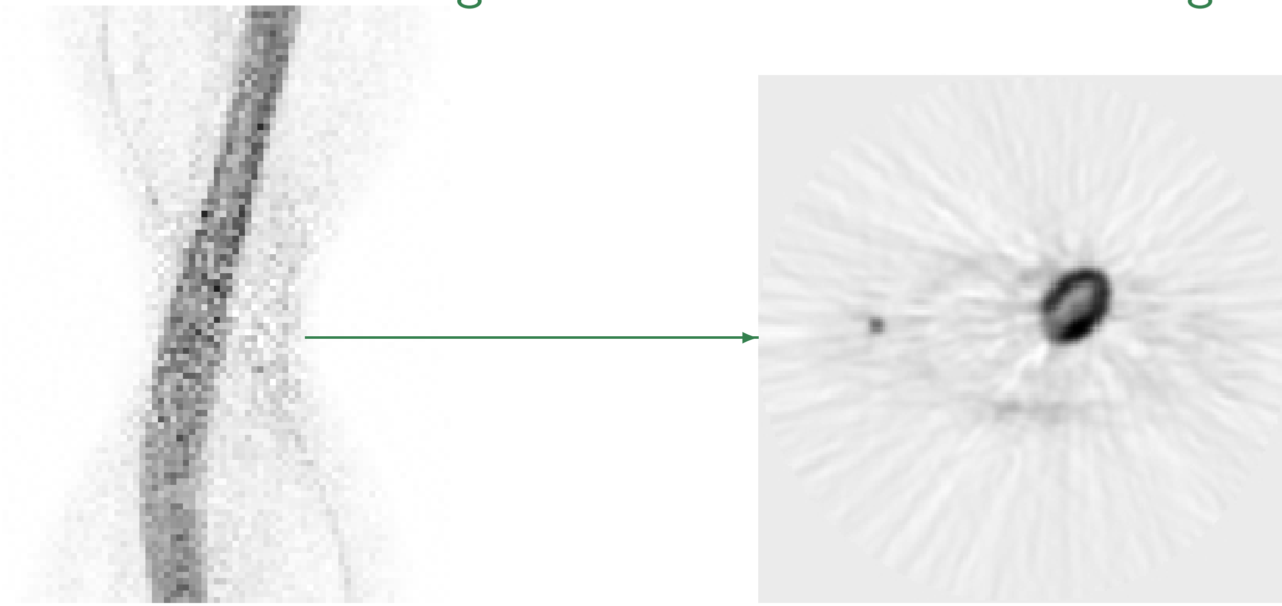


Figure 1: Image reconstruction from projections (negative images)

- The image is estimated computationally (inverse problem).

- Different tomographic modalities reflect different things.

Figure 2: X-ray based CT (anatomical)

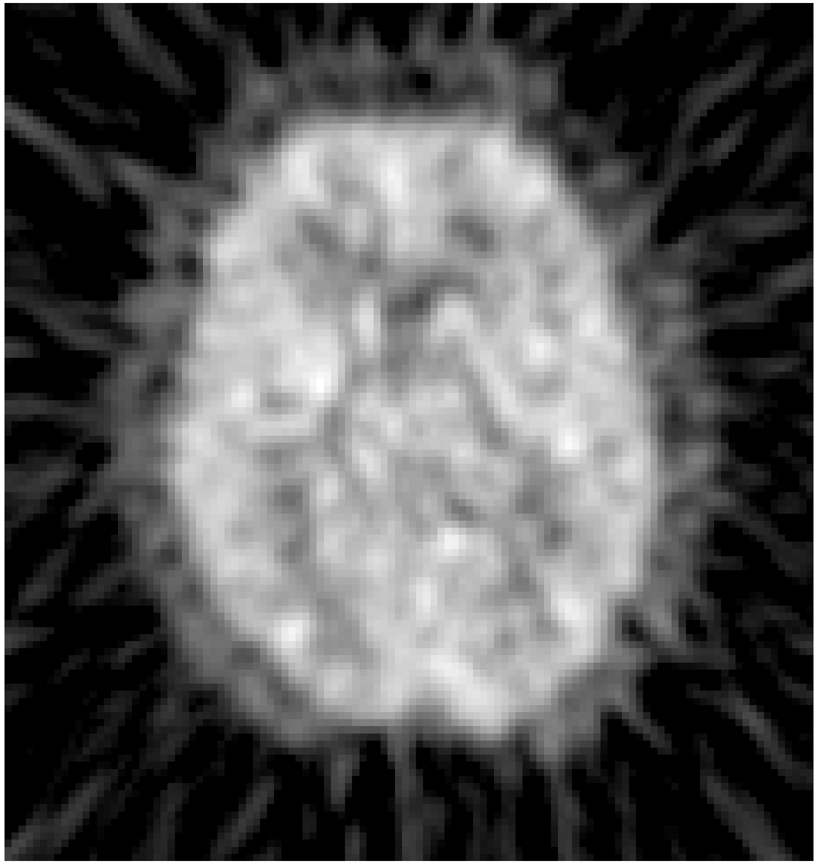


Figure 3: PET (functional)

Radon transform

- The tomographic data acquisition is conventionally modeled by the **Radon transform** (Johann Radon, 1917).

- Radon transform collects line integrals across the object at different angles.

$$m(t,\theta) \triangleq \mathcal{R}\{f\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - t)\,dx\,dy\ . \qquad (1)$$

- Note: Radon $m(t,\theta)$ is *not* a polar coordinate representation.

- Measured data are collected as a **sinogram** matrix.

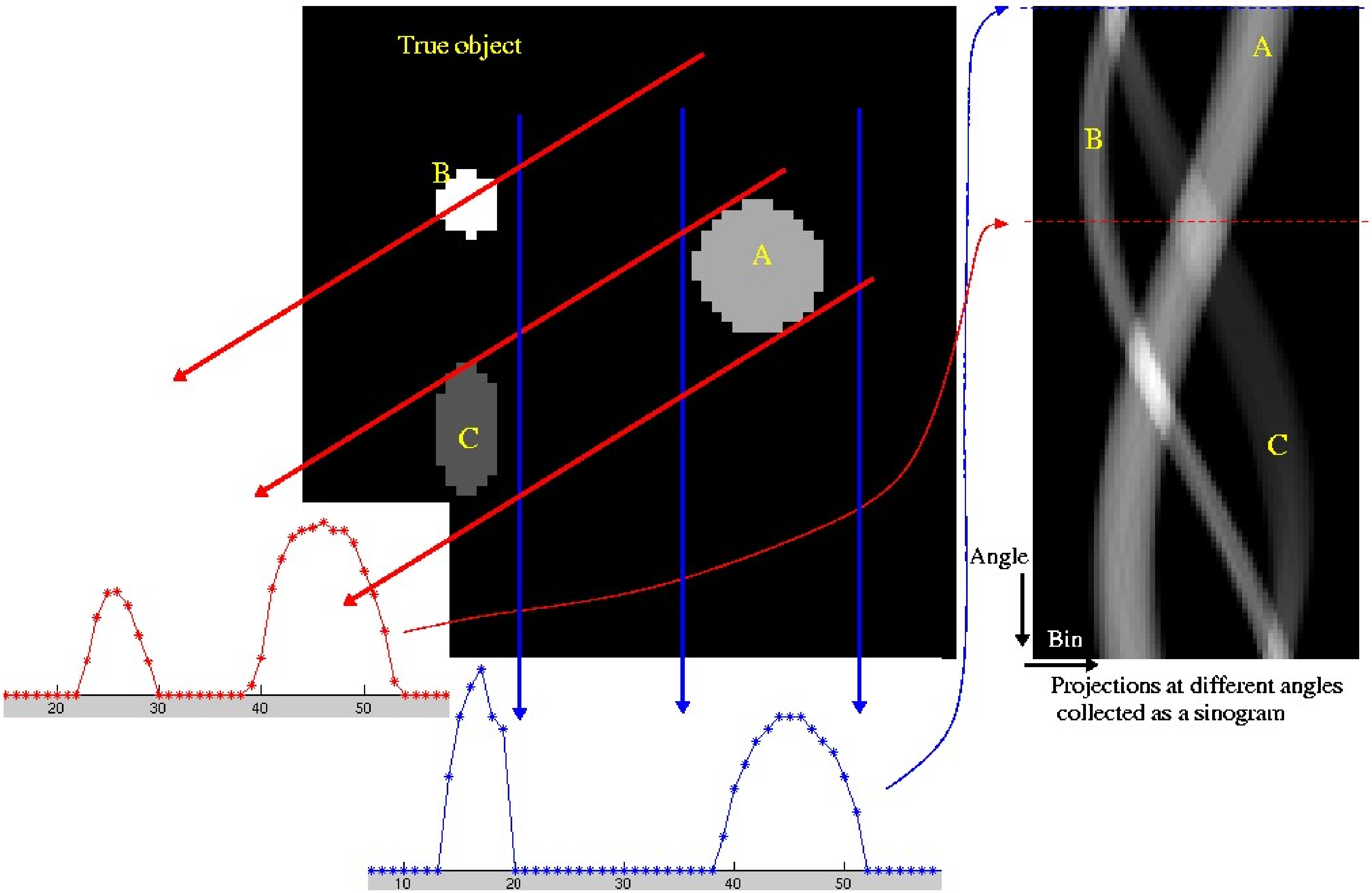


Figure 4: Projection data collected as a sinogram (Radon transform of the unknown object).

Fourier slice theorem

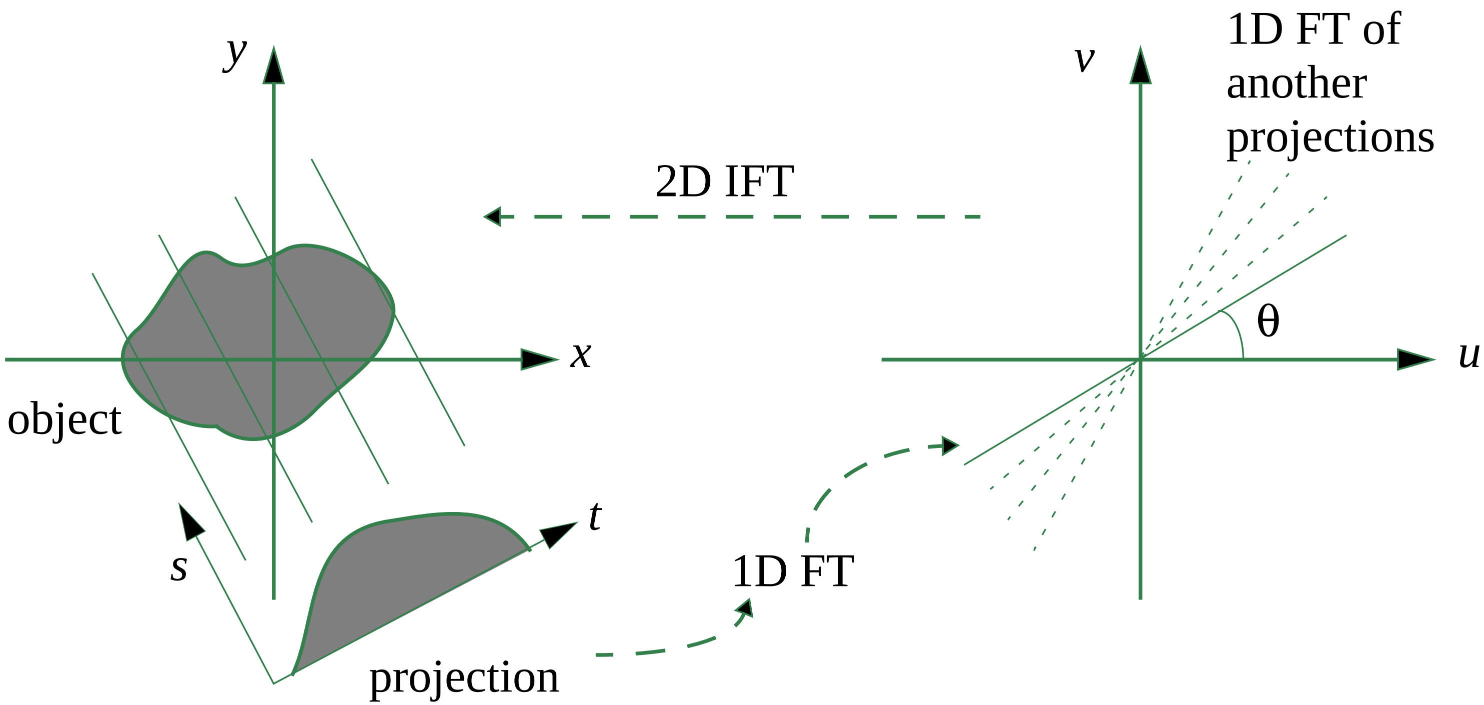


Figure 5: The Fourier slice theorem: The 1D FT of a projection taken at angle θ equals the central radial slice at angle θ of the 2D FT of the original object.

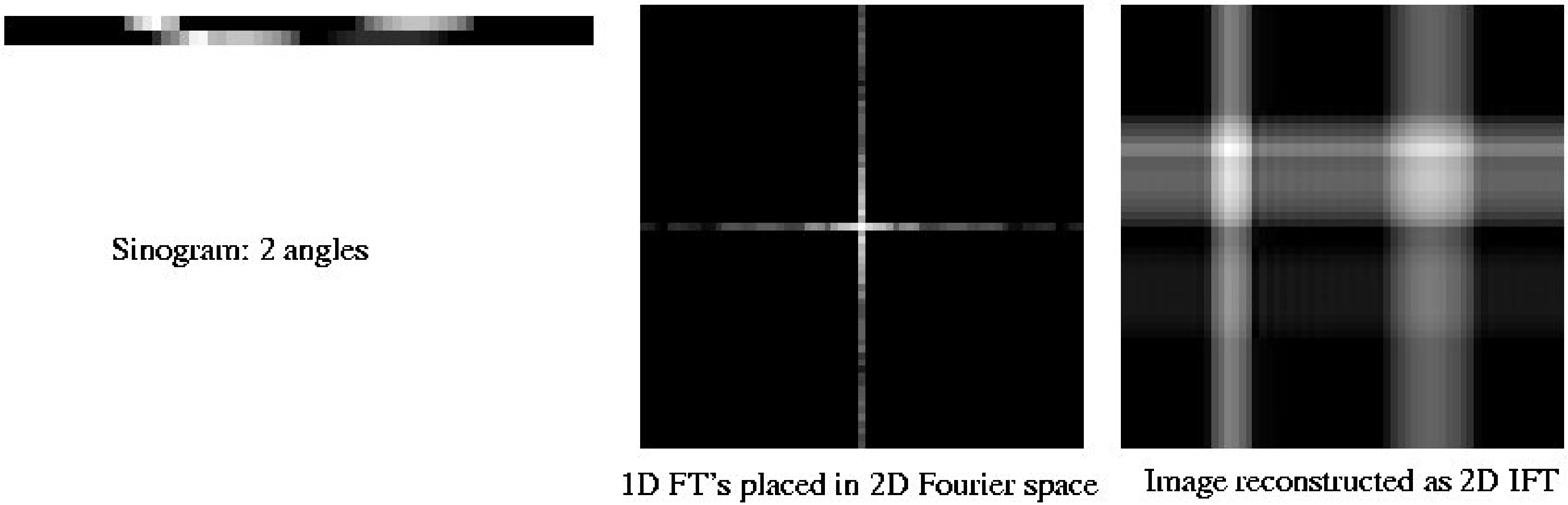


Figure 6: 2 projections reconstructed using simple direct Fourier method.

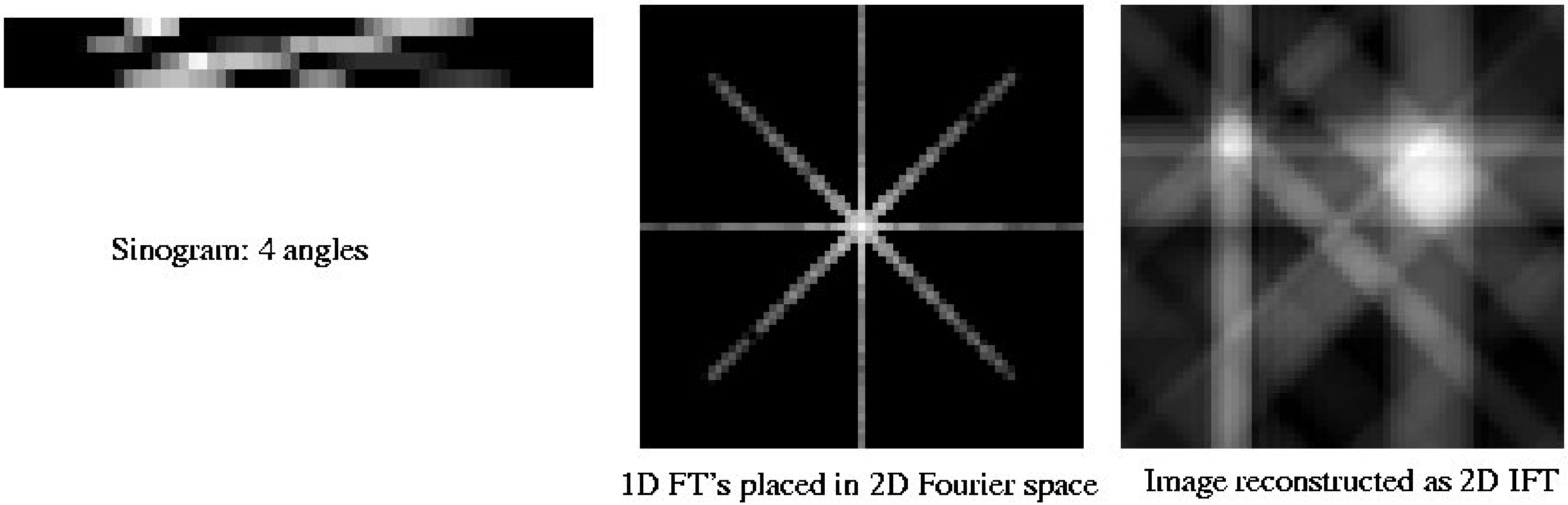
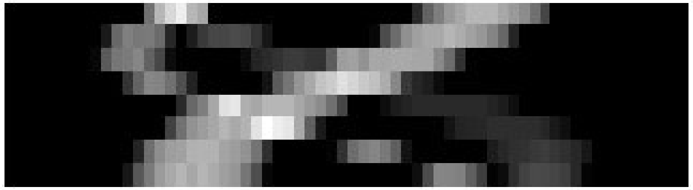
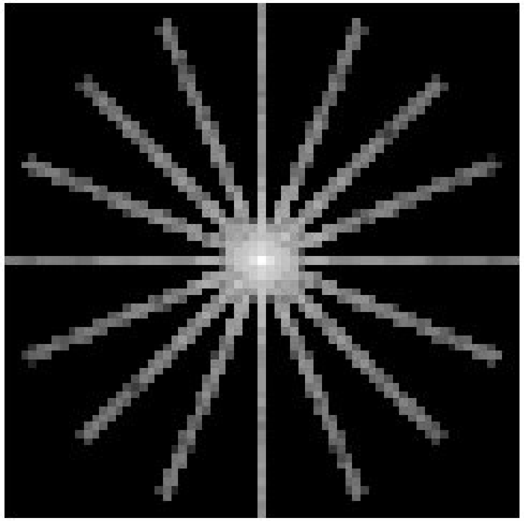


Figure 7: 4 projections reconstructed using simple direct Fourier method.



Sinogram: 8 angles



1D FT's placed in 2D Fourier space

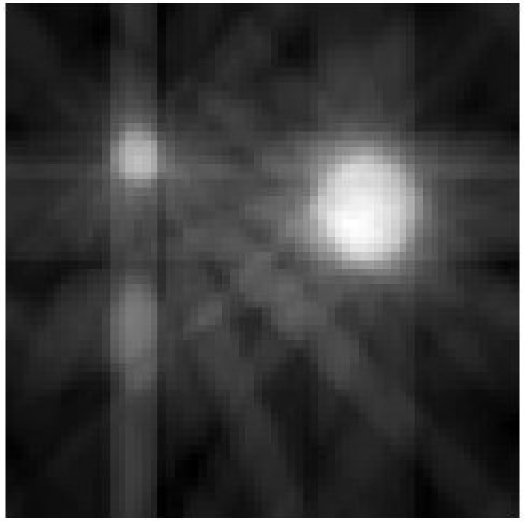


Image reconstructed as 2D IFT

Figure 8: 8 projections reconstructed using simple direct Fourier method.

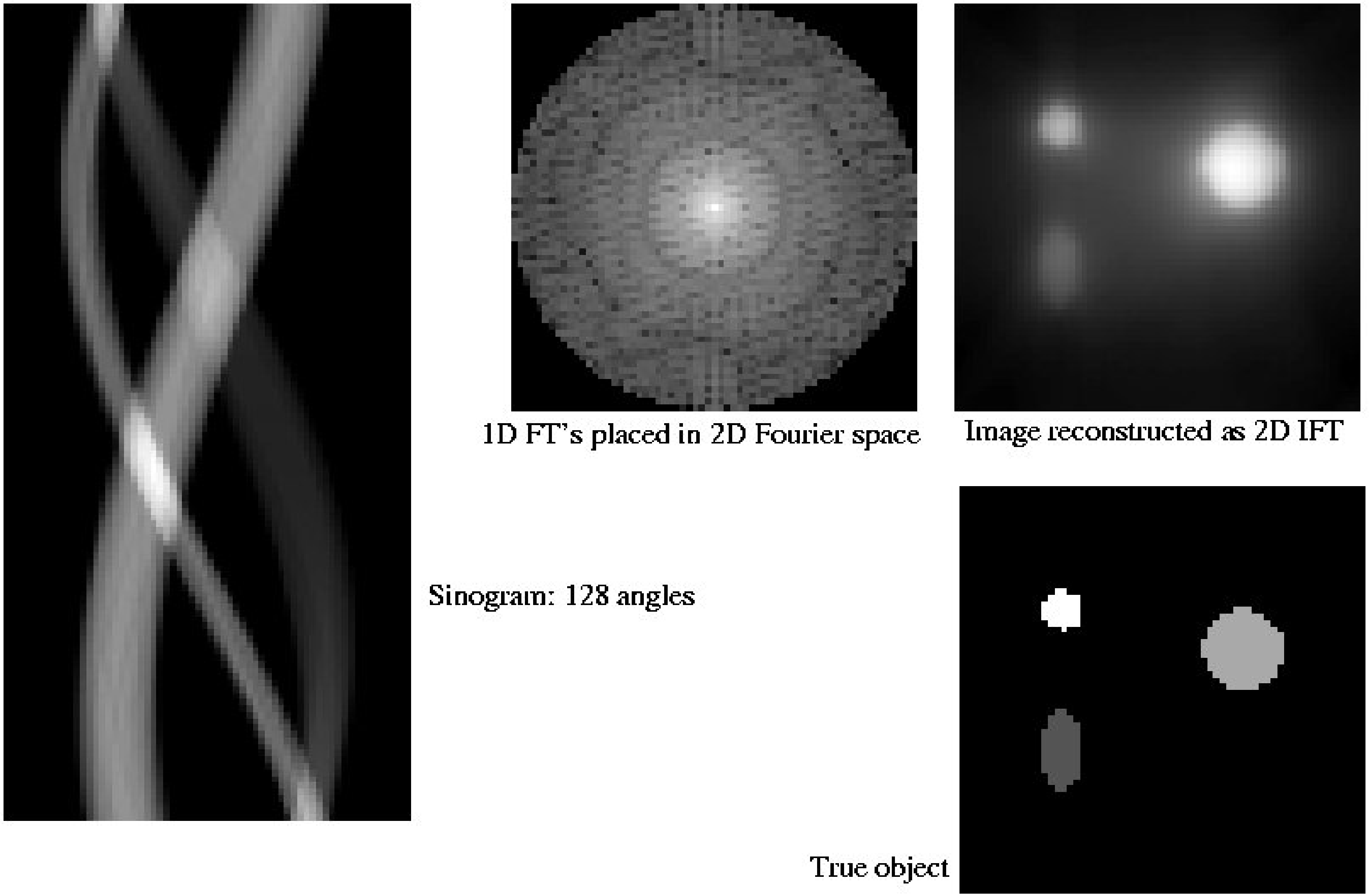


Figure 9: 128 projection views reconstructed using simple direct Fourier method.

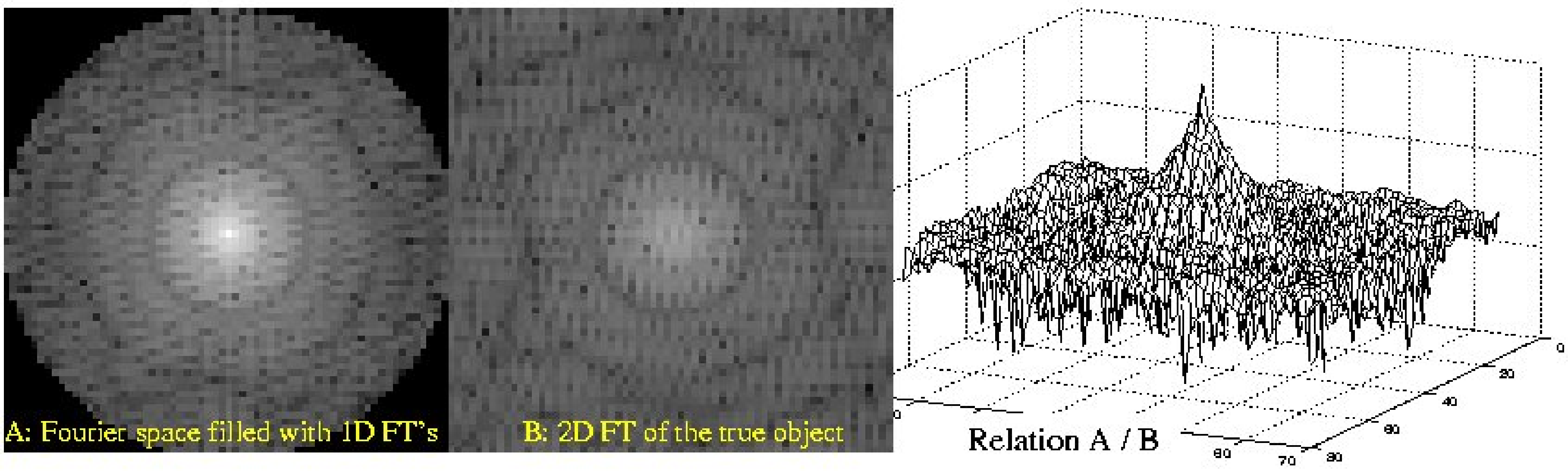


Figure 10: The density difference btw. direct Fourier and FT of the original image.

- The Fourier space filled in is most dense at and near the zero frequency.
- Compensation by the distance from the center $|\omega| \longrightarrow$ Ramp-shaped filter.
- Interpolation errors in the corners (high frequencies!) make direct application of Fourier slice theorem difficult $\longrightarrow$ *not used* in practice

Inverse Radon transform by FBP

- The inverse 2D FT expressed using the polar coordinates ω and θ in the frequency space $(u = \omega \cos\theta, v = \omega \sin\theta)$ is

$$
\begin{aligned}
f(x,y) &= \mathcal{F}_2^{-1}\{F(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(xu+yv)}\, du\, dv \\
&= \int_0^{2\pi}\int_0^{\infty} F(\omega\cos\theta, \omega\sin\theta)\, e^{j2\pi\omega(x\cos\theta + y\sin\theta)} \underbrace{\begin{vmatrix} \frac{\partial u}{\partial \omega} & \frac{\partial v}{\partial \omega} \\ \frac{\partial u}{\partial \theta} & \frac{\partial v}{\partial \theta} \end{vmatrix}}_{=\omega} d\omega\, d\theta \\
&= \int_0^{\pi} \left[\int_{-\infty}^{\infty} M(\omega,\theta)|\omega|\, e^{j2\pi\omega(x\cos\theta + y\sin\theta)}\, d\omega \right] d\theta\ , \qquad (2)
\end{aligned}
$$

where $M(\omega, \theta)$ is the 1D FT of the measure projection profile $m(t, \theta)$.

- The multiplication by $|\omega|$ serves as a ramp filter applied to each projection profile in the frequency space.

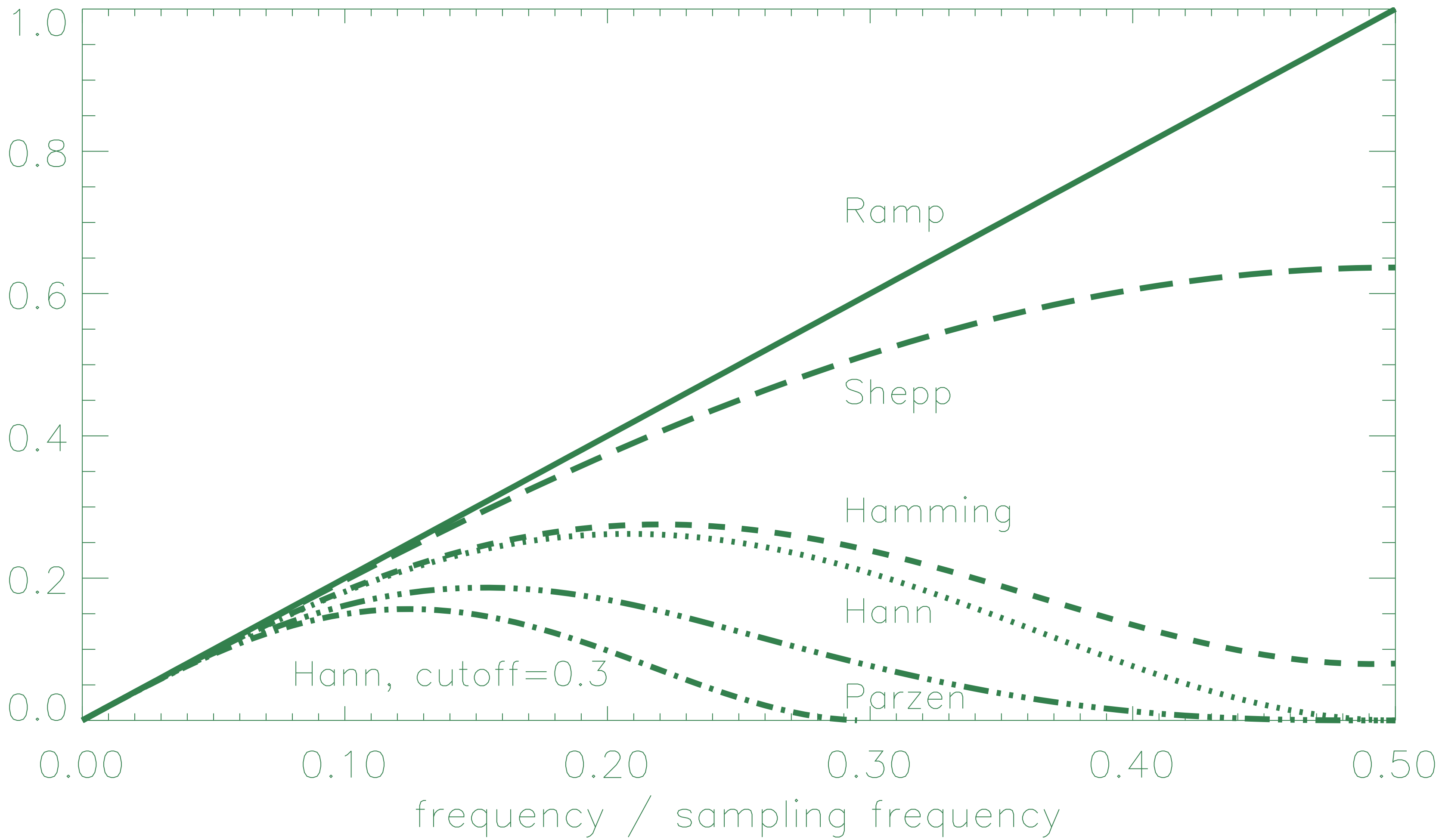


Figure 11: Common window functions used with ramp filter.

- (2) gives an algorithm to reconstruct the image $f(x, y)$ from its projections $m(t, \theta)$ as the **Filtered Back Projection, FBP**: Set $f(x, y) = 0, \forall x, y$. For each projection profile:

  - ⋆ Take 1D FT: $m(t, \theta) \longrightarrow M(\omega, \theta)$
  - ⋆ Apply the frequency domain filter (e.g. ramp + Hann)
  - ⋆ Take inverse FT: $|\omega| M(\omega, \theta) \longrightarrow \hat{m}(t, \theta)$
  - ⋆ Back project (smear) filtered profile $\hat{m}$ over the image at the given angle θ

- In the discrete implementation of FBP, the integrals are replaced by finite summations and FFTs can be used.

Figure 12: FBP with 2 projections (with noise)

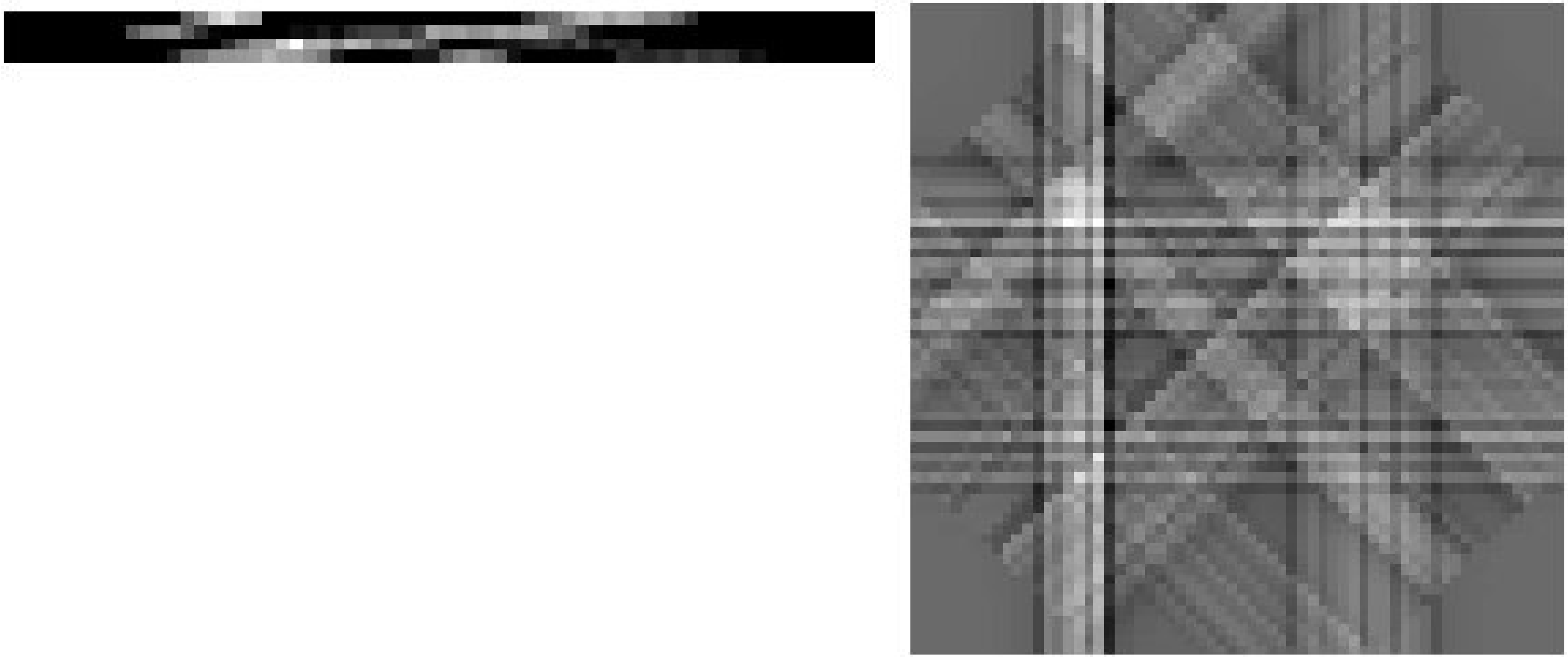


Figure 13: FBP with 4 projections

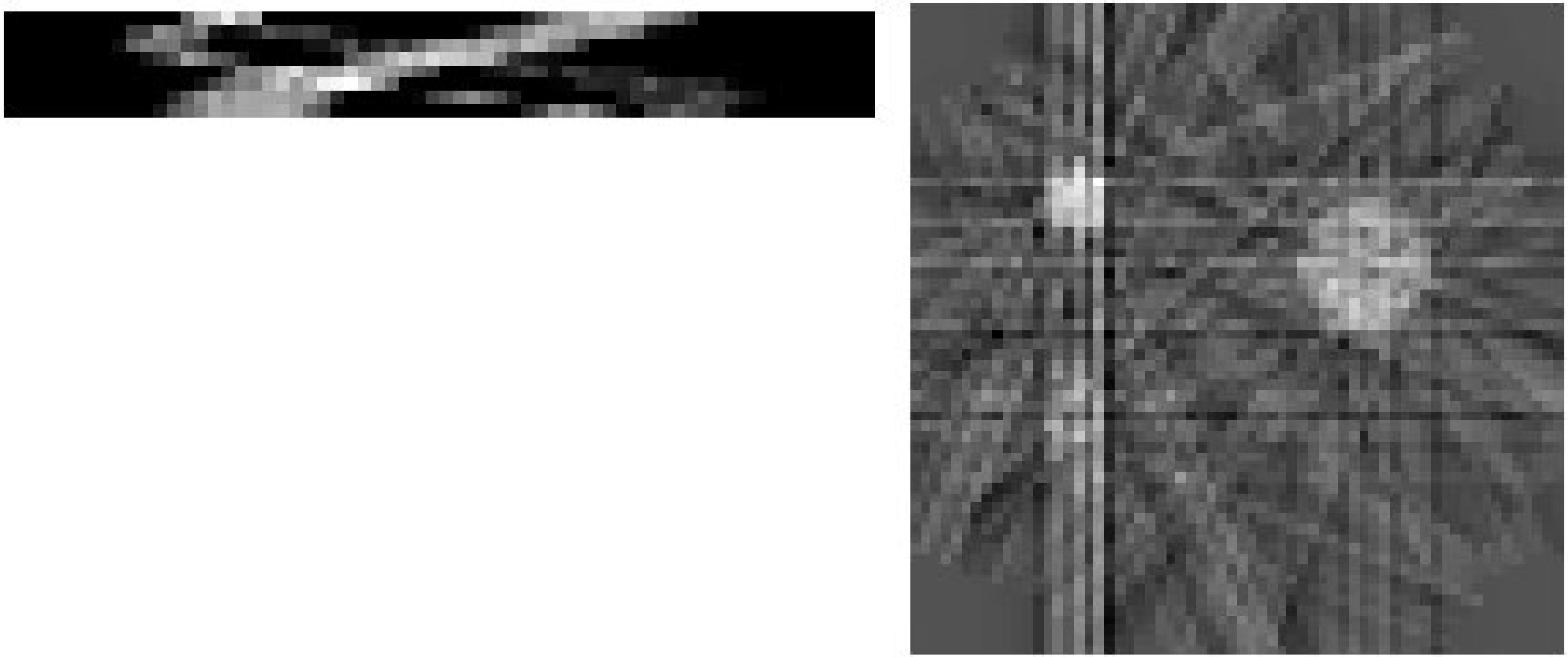


Figure 14: FBP with 8 projections

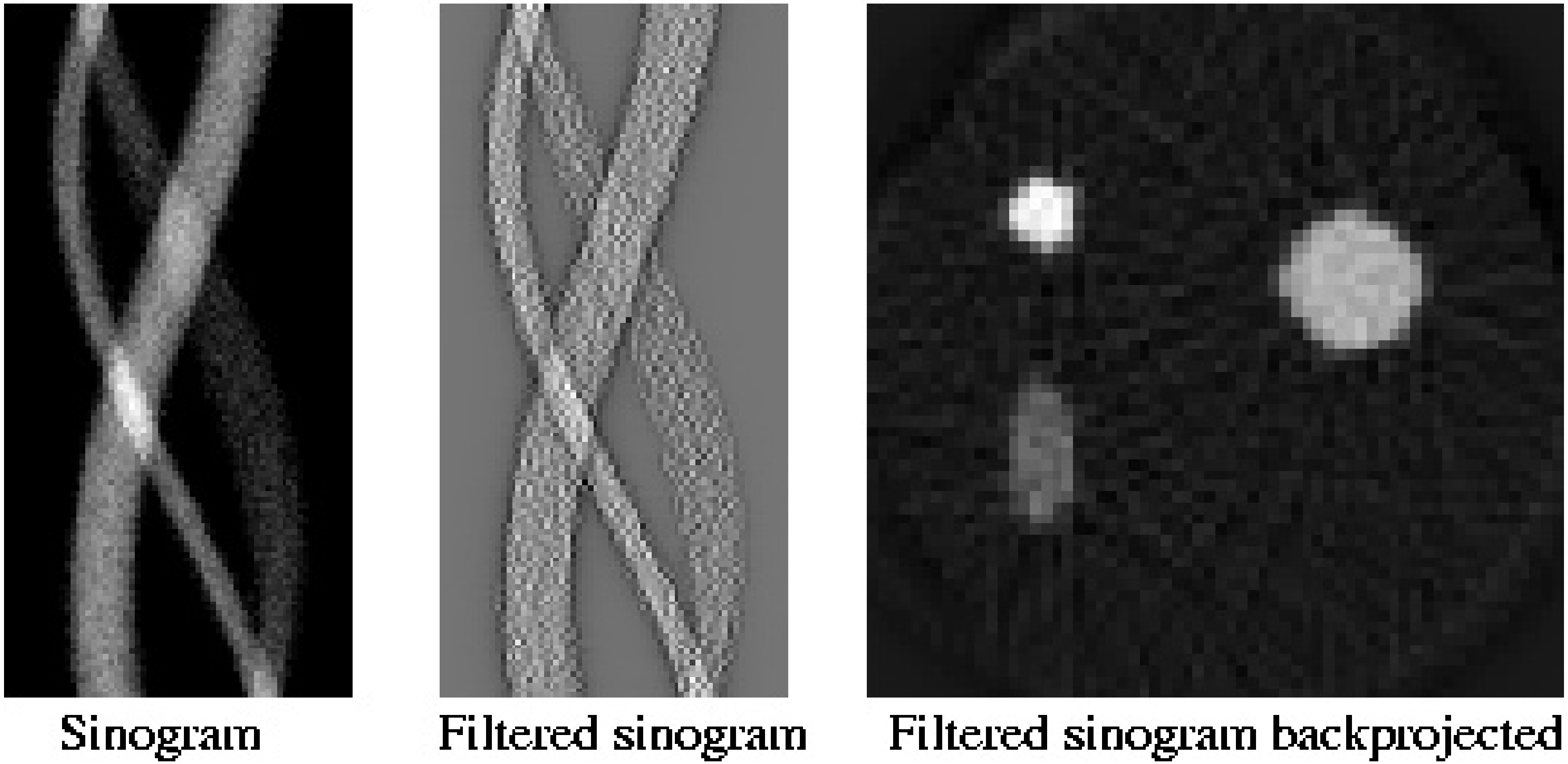


Figure 15: FBP with 128 projections

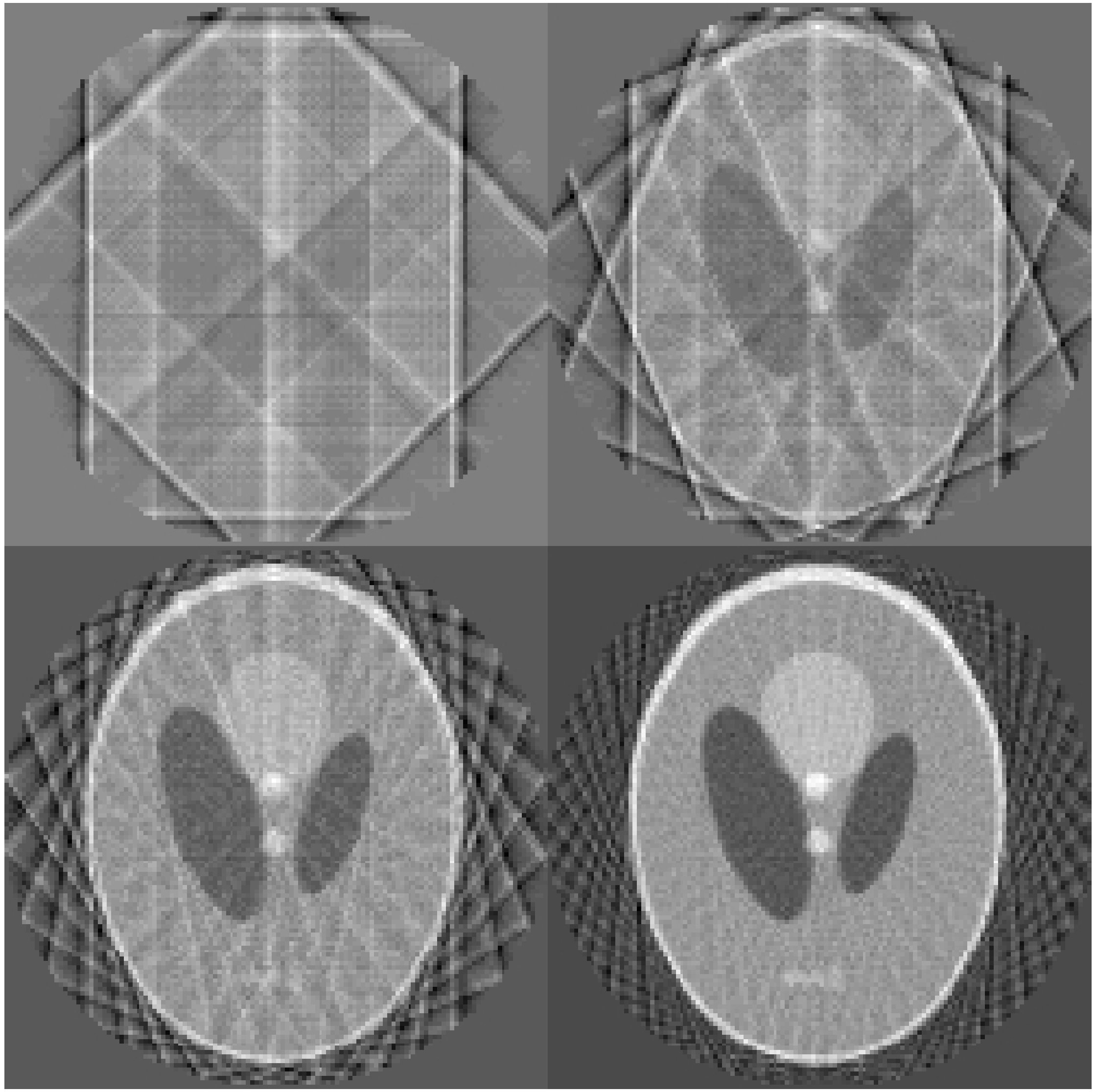


Figure 16: Number of projection angles: 4, 8, 16, 32 views

Iterative reconstruction

- Iterative reconstruction: the reconstructed image is a solution of a maximization of an objective function.

- Starting from an initial guess, the image is updated iteratively so that it matches better the measured projections.

- Maximum likelihood expectation maximization (MLEM) searches for an image that makes the measured data most likely to occur $(argmax_\lambda p(n|\lambda))$

$$\lambda_b^{\langle k+1 \rangle} = \frac{\lambda_b^{\langle k \rangle}}{\sum_d p_{db}} \sum_d \frac{n_d \; p_{db}}{\sum_{b'} \lambda_{b'}^{\langle k \rangle} p_{db'}} \qquad (3)$$

  $\lambda^{\langle k \rangle}$ is the kth iteration emission image, b is the pixel index, d is the sinogram bin index, p_{db} is a system matrix element, and $\boldsymbol{n}$ is the measured sinogram.

- With noisy data, the reconstructed image is also noisy.

- ART (algebraic reconstruction technique): The update is additive. The idea is to solve a set of linear equations

$$\boldsymbol{n} = \boldsymbol{A}\boldsymbol{\lambda} \tag{4}$$

(${\boldsymbol{n}}$ sinogram, $\boldsymbol{A}$ system matrix, $\boldsymbol{\lambda}$ image) and update the image according to the difference btw. calculated and measured projections.

- Other similar methods: SIRT, SART.

- Due to noise, no unique solution exists.

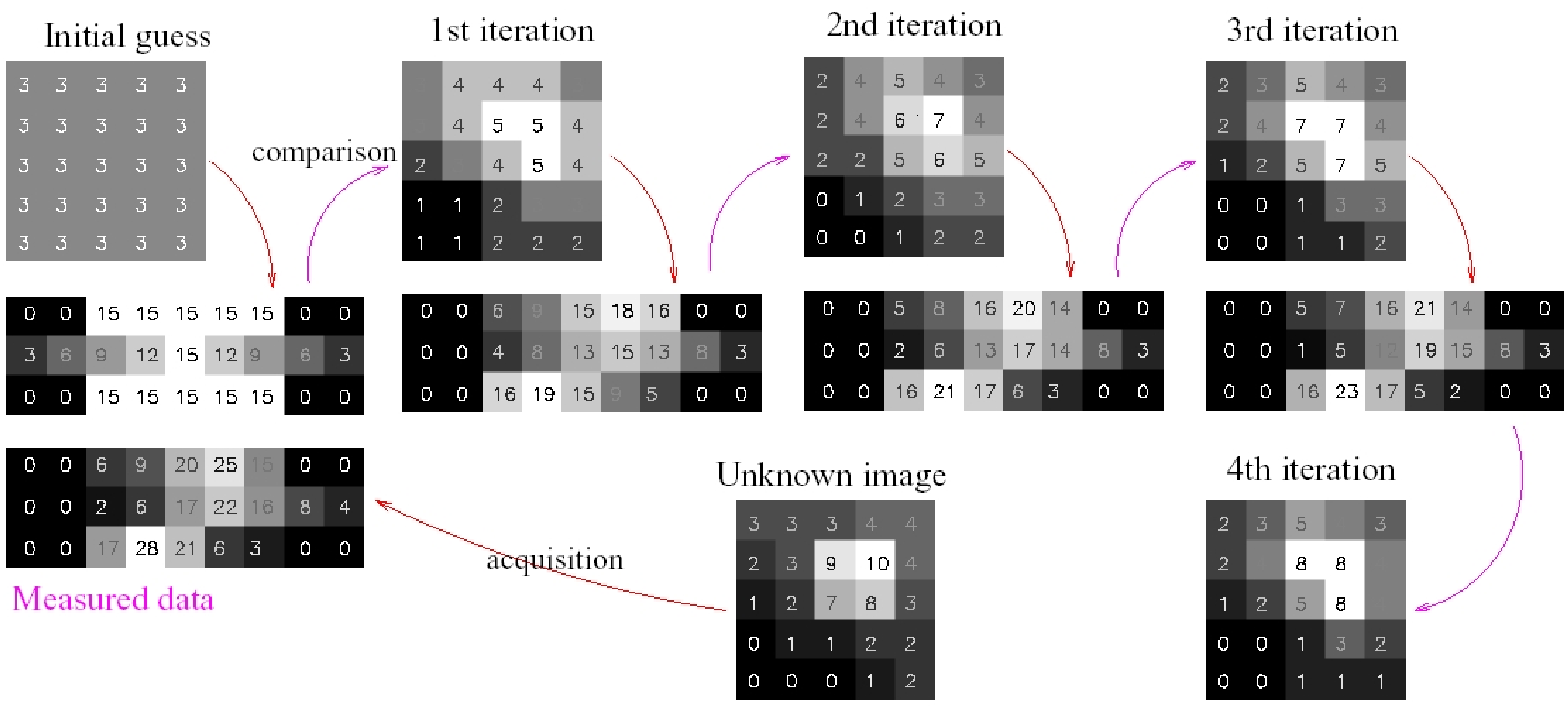


Figure 17: Simple example of MLEM iterations

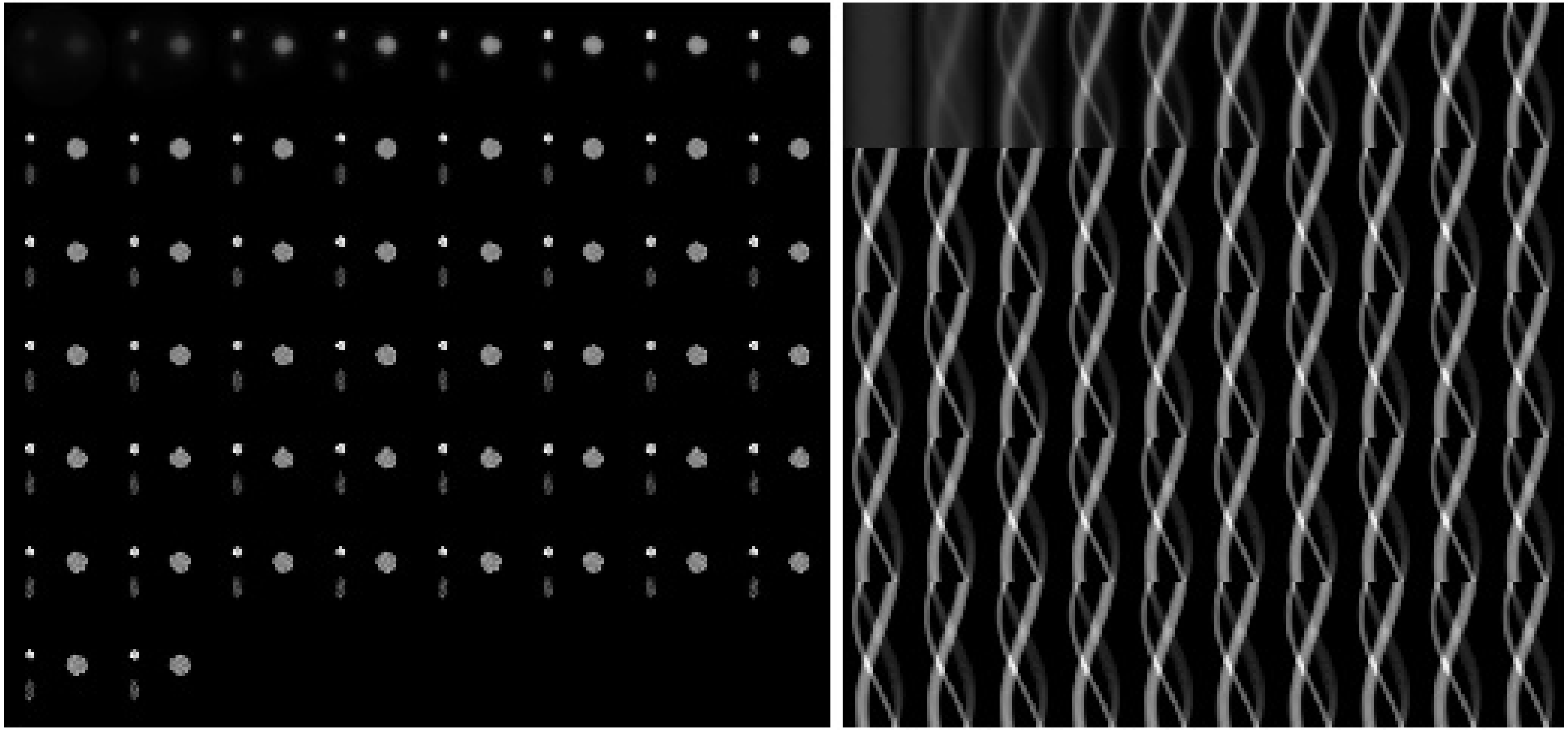


Figure 18: Intermediate images and reprojections of MLEM iterations

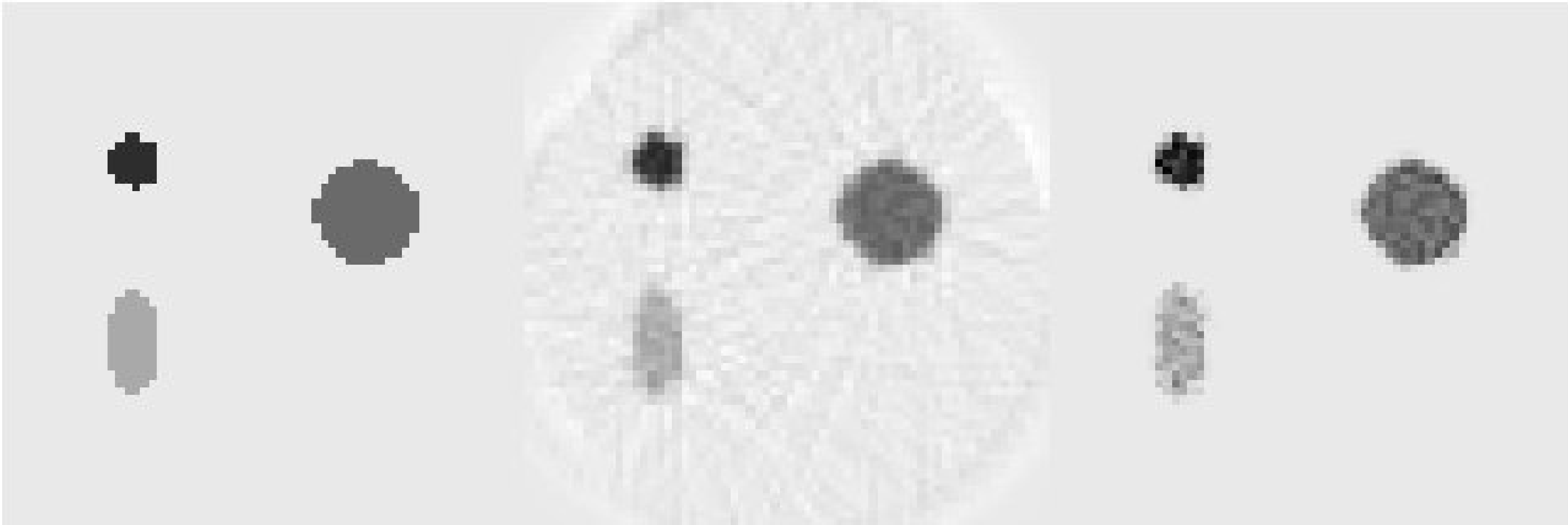


Figure 19: True, FBP, MLEM (with noise)

Penalized iterative reconstruction

- Ill-posed to well-posed: The image is required to fit with measured data, and also be consistent with additional regularizing criteria that are set independent on the data.

- The one step late (OSL) algorithm uses the current image $\boldsymbol{\lambda}^{\langle k \rangle}$ when calculating the value of the derivative of the energy function $U()$.

$$\lambda_b^{\langle k+1 \rangle} = \frac{\lambda_b^{\langle k \rangle}}{\sum_d p_{db} + \beta \frac{\partial}{\partial \lambda_b} U(\boldsymbol{\lambda}, b) \,|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^{\langle k \rangle}}} \underbrace{\sum_d \frac{n_d \, p_{db}}{\sum_{b'} \lambda_{b'}^{\langle k \rangle} p_{db'}}}_{c_b^{L\langle k \rangle}}$$

$$= \lambda_b^{\langle k \rangle} c_b^{P\langle k \rangle} c_b^{L\langle k \rangle} \qquad (5)$$

$\lambda^{\langle k \rangle}$ is the kth iteration emission image, b is the pixel index, d is the sinogram bin index, p_{db} is a system matrix element, and $\boldsymbol{n}$ is the measured sinogram.

- The current image estimate $\lambda^{\langle k \rangle}$ is updated using two factors: $c_b^{P\langle k \rangle}$ changes the pixel value such that prior assumptions are better met, and $c_b^{L\langle k \rangle}$ for better data fitting (MLEM).

- The penalty can be restricted to only non-monotonic structures in a neighborhood by comparing the pixel against the local median.

  Using this constraint in the term $U()$ of Eq. (5), the penalty factor $c_b^{P\langle k \rangle}$ for MRP (median root prior) is

$$c_b^{P\langle k \rangle} = \frac{1}{\sum_d p_{db} + \beta \, \frac{\lambda_b^{\langle k \rangle} - M_b}{M_b}} \tag{6}$$

  The penalty reference $M_b = Med(\lambda^{\langle k \rangle}; b)$ is estimated as the median of the pixels in a neighborhood N_b

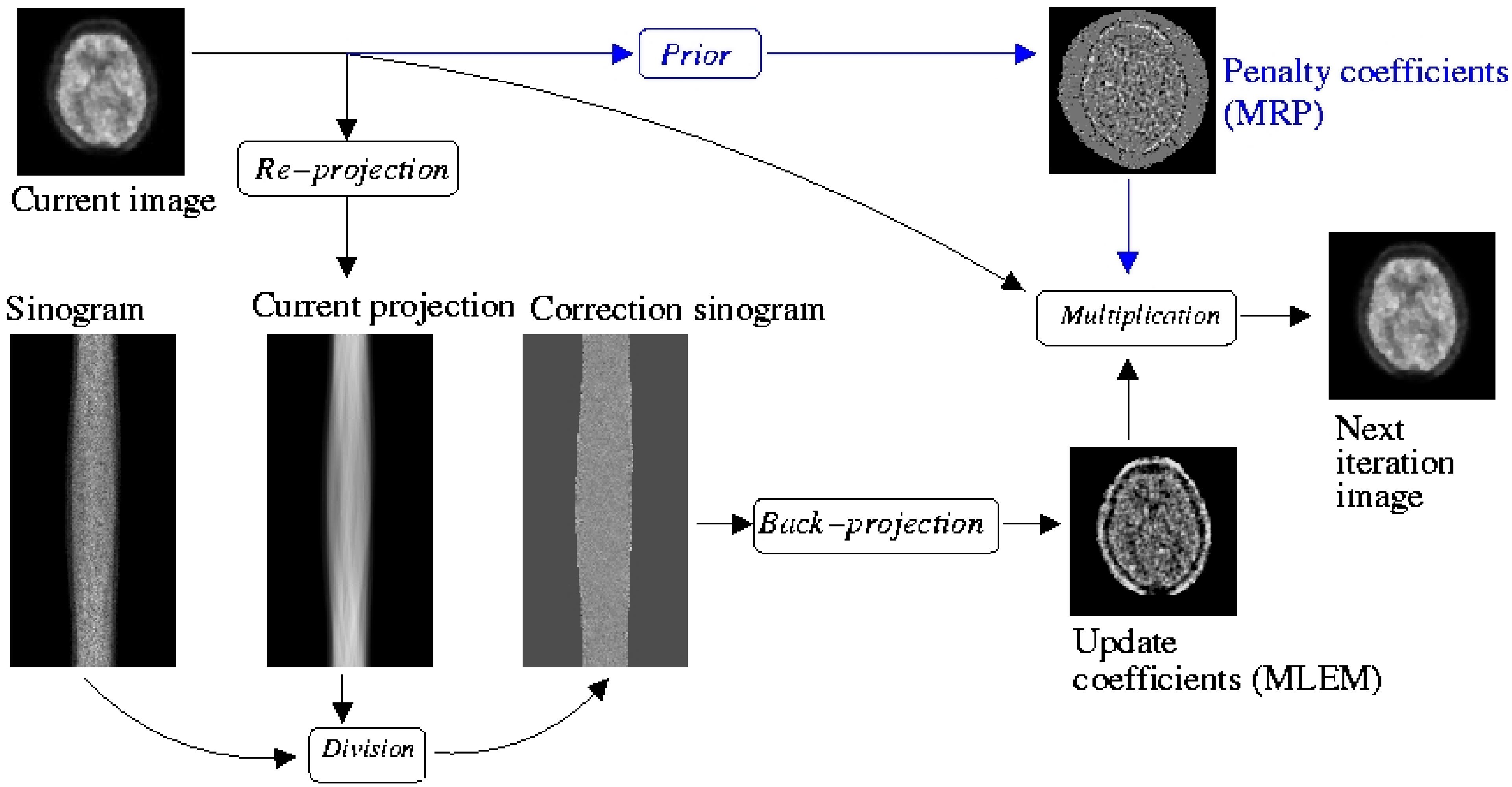


Figure 20: Flowchart of penalized MLEM reconstruction

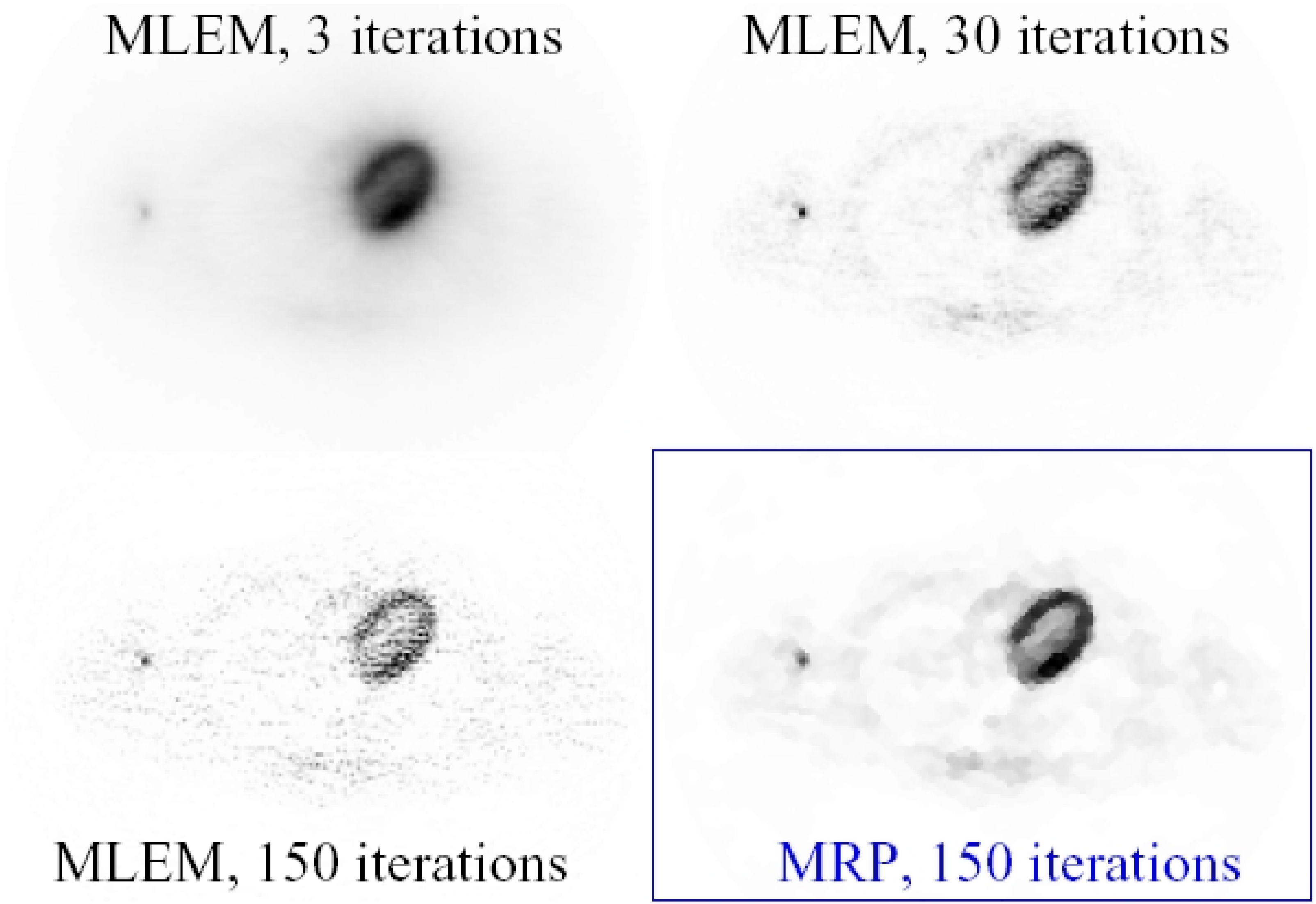


Figure 21: Large number of iterations makes MLEM-image more noisy. MRP is more robust.

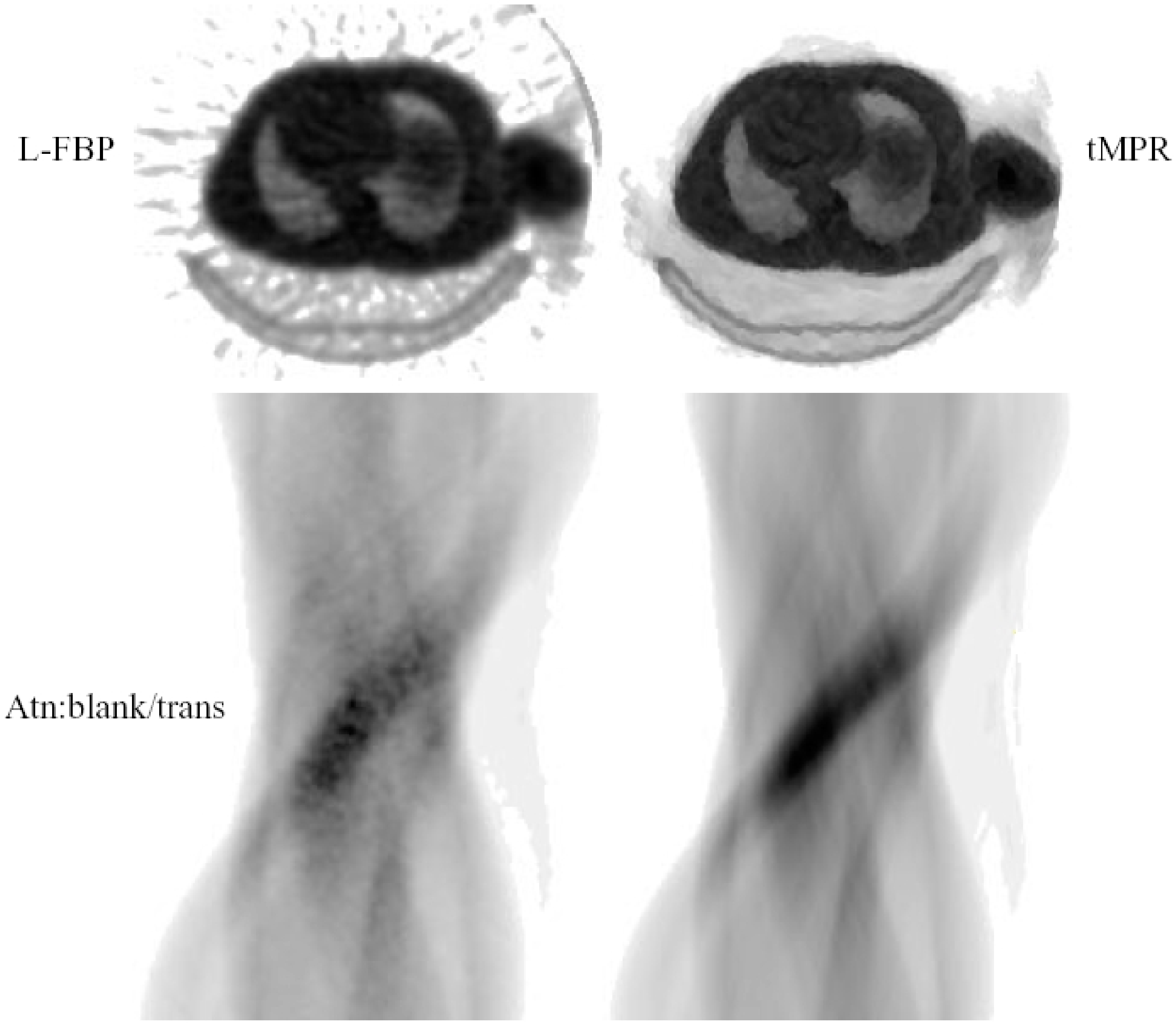


Figure 22: Transmission scans are used in PET for attenuation compensation.

Conclusion

- FBP works if data are not very noisy and if the measurement can be accurately modeled as a Radon transform.
- Noise regularization is frequency selective (cutoff & window) $\longrightarrow$ trade-off btw. resolution and noise.
- Iterative methods maximize an objective function $\longrightarrow$ A probabilistic noise model can be used.
- Quantitative accuracy is important in some studies (PET) $\longrightarrow$ noise reduction should be unbiased.
- Noise regularization: early stopping, priors & penalties.
- MRP is effective, but some theoretical aspects are lost (no proof of convergence).

- Noise reduction directly from the sinogram is a demanding task. Some progress done already.

- Some online links:
  - ⋆ Kak, Slaney: Principles of Computerized Tomographic Imaging http://www.slaney.org/pct/
  - ⋆ Let's play PET http://laxmi.nuc.ucla.edu:8000/lpp/
  - ⋆ Bruyant: Analytic and Iterative Reconstruction Algorithms in SPECT http://www.snm.org/education/pq1002.html, http://jnm.snmjournals.org/cgi/content/full/43/10/1343
  - ⋆ MIRG's SPECT Tutorial http://www.physics.ubc.ca/~mirg/home/tutorial/fbp_recon.html